Infinite-dimensional Noether symmetry groups and quantum effective actions from geometry

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We consider quantum effective actions for arbitrary models possessing an infinite-dimensional group G of Noether symmetries. The relevant Ward identities yield functional differential equations for the effective action whose exact solution is found to be given by the geometric action on a coadjoint orbit of the (central extended) Noether group $\tilde{G}$. As a particular application we show that the effective action of the light-cone quantized toroidal membrane is explicitly given by the geometric co-orbit action of the group of area-preserving diffeomorphisms on torus.

1. Introduction

Classical- and quantum-mechanical systems with infinite-dimensional groups of Noether symmetries are attracting broad interest since a couple of years. There emerged so far three principal classes of such systems:

(i) Completely integrable models in $D=2$ space–time dimensions [1] (the relevant symmetries form an infinite-dimensional abelian group);

(ii) $D=2$ (super-)conformal field theories [2] with their fundamental applications in (super)string theory [3] (Kac–Moody and Virasoro symmetries and their supersymmetric extensions);

(iii) theory of (super) p-branes [4] (the relevant symmetries being groups of symplectic (volume-preserving) diffeomorphisms).

In (i), the presence of the infinite-dimensional abelian symmetry is exploited in the powerful classical and quantum inverse scattering method [1] leading to exact solutions. In (ii) and (iii) the relevant symmetries are of different nature and non-abelian, but still they determine the whole dynamics of the systems. One of the most efficient ways to incorporate their infinite-dimensional symmetry structure is the method of group coadjoint orbits [#1].

In the present letter, starting from the general symplectic manifold's formalism (generalizing the group coadjoint orbit formalism; see section 2) we discuss arbitrary models (in $D\geqslant 2$ dimensions) possessing an infinite-dimensional group G of Noether symmetries and consider coupling of the Noether currents to external "sources". We write down the Ward identities for the corresponding quantum effective actions which turn out to exhibit the following remarkable property (section 3). No matter what is the specific action of the initial classical model, its quantum effective action is always given by the geometric action on a generic coadjoint orbit of the (central extension of the) Noether symmetry group $\tilde{G}$. Given therefore the model with Noether symmetry algebra which admits central extension, the geometric action ap-

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[#1] Some basic references for the recent revival of the coadjoint orbit method [5] in the context of its extension to infinite-dimensional groups and applications to $D=2$ conformal field theories are [6–9].

proach [10–12] can then be employed to find its unique quantum effective action.

The last, fourth section is devoted to specific examples of this general result.

2. General symplectic actions

Let us consider an infinite-dimensional symplectic manifold (phase space) $\mathcal{M}_S$ parametrized by local coordinates S^i. The index i is a short-hand notation for

$$i=((x_1,...,x_p);A)\,, \tag{1}$$

including in general both continuous parameters (x_1, ..., x_p) (e.g., in the case of p-brane models) as well as discrete indices A (as in the case of Kac–Moody groups). The fundamental Poisson brackets (PB) are of the form

$$\{S^i,S^j\}_{\rm PB}=\omega^{ij}(S)\,, \tag{2}$$

$$\omega^{ij}(S)=\overset{0}{\omega}{}^{ij}+\overset{0}{\omega}{}^{ij}_{k}S^k+\tfrac{1}{2}\overset{0}{\omega}{}^{ij}_{kl}S^kS^l+\ldots\,. \tag{3}$$

We make in the following an assumption that the coordinate system $\{S^i\}$ of $\mathcal{M}_S$ is such that the PB matrix $\omega^{ij}(S)$ is invertible on the whole manifold $\mathcal{M}_S$. In what follows, we shall keep for simplicity the series (3) only up to second order in S^i [#2]. The Jacobi identities for the Poisson bracket (2) read

$$\sum_{i,j,k}^{\rm cyclic}\omega^{il}(S)\frac{\delta}{\delta S^l}\omega^{jk}(S)=0\,, \tag{4}$$

from which one easily deduces a number of fundamental identities for the coefficients in (3).

Let us also introduce a one-form $Y_i=Y_i(S)$ on $\mathcal{M}_S$ as a (non-local and non-linear) functional of S^i determined from the following basic equation:

$$\mathrm{d}S^i+\omega^{ij}(S)Y_j=0\,, \tag{5}$$

where d is the exterior derivative on $\mathcal{M}_S$. The integrability condition for (5) together with (4) imply that Y_i must satisfy the generalized Maurer–Cartan equation:

$$\mathrm{d}Y_i+\frac{1}{2}\left(\frac{\delta}{\delta S^i}\omega^{kl}(S)\right)Y_k\wedge Y_l=0\,. \tag{6}$$

Before proceeding further it is useful to comment, at this point, on the explicit nature of the objects introduced above in the important particular case when the symplectic manifold $\mathcal{M}_S$ is (locally) isomorphic to a coadjoint orbit $\mathcal{O}_{(B_0,c)}$ of the (central-extended) Lie group $\tilde{\mathrm{G}}$, with a Lie algebra $\tilde{\mathcal{G}}=\mathcal{G}+\mathbb{R}$, passing through a generic point $(B_0,\,c)$ in the dual space $\tilde{\mathcal{G}}^*=\mathcal{G}^*+\mathbb{R}$. In this case the PBs (2) acquire the form (i.e. $\overset{0}{\omega}{}^{ij}_{kl}=0$)

$$\{S^i,S_j\}=\overset{0}{\omega}{}^{ij}+\overset{0}{\omega}{}^{ij}_{k}S^k\,. \tag{7}$$

Then eq. (4) reduces to the usual Jacobi identities for the structure constants $\overset{0}{\omega}{}^{ij}_{k}$ and the condition that $\overset{0}{\omega}{}^{ij}$ is a (non-trivial) $\mathcal{G}$-cocycle (defining the central extension from $\mathcal{G}$ to $\tilde{\mathcal{G}}$).

Let us choose a basis $(T^i,\hat{E})$ in $\tilde{\mathcal{G}}$:

$$[T^i,T^j]=-\overset{0}{\omega}{}^{ij}_{k}T^k-\overset{0}{\omega}{}^{ij}\hat{E},\quad [T_i,\hat{E}]=0\,, \tag{8}$$

with the associated dual basis $(T_i^*,\,\hat{E}^*)$ in $\tilde{\mathcal{G}}^*$ (i.e. $\langle T_i^*|T^j\rangle=\delta_i^j$ where $\langle\ |\ \rangle$ is the natural bilinear form on $\mathcal{G}^*\times\mathcal{G}$). Let us also introduce the following functions on the group G (the non-centrally extended part of $\tilde{\mathrm{G}}$) corresponding to the algebra $\mathcal{G}$: $\mathcal{G}$:

$$\Sigma(g)\equiv S^iT_i^*\in\mathcal{G}^*,\quad Y(g)\equiv Y_iT^i\in\mathcal{G}\,. \tag{9}$$

Then, the coadjoint action of $\tilde{\mathrm{G}}$ on $\tilde{\mathcal{G}}^*$ becomes (see refs. [5,12])

$$\widetilde{\mathrm{Ad}}^*(g)(B_0,c)=(B_0+\Sigma(g),c)\,, \tag{10}$$

and the coadjoint orbit is parametrized in terms of the group coordinates g as

$$\mathcal{O}_{(B_0,c)}=\{(B_0+\Sigma(g),c);\ \forall g\in\mathrm{G}\}\,. \tag{11}$$

Eqs. (5) and (6) can be rewritten as

$$\mathrm{d}\Sigma(g)=\mathrm{ad}^*(Y(g))\Sigma(g)+\sigma(Y(g))\,, \tag{12}$$

$$\mathrm{d}Y(g)=\tfrac{1}{2}[Y(g),\wedge Y(g)]\,, \tag{13}$$

where $\mathrm{ad}^*(\ )$ is the Lie-algebra $\mathcal{G}$ coadjoint action on $\mathcal{G}^*$:

$$\langle\mathrm{ad}^*(\xi)B|\eta\rangle=-\langle B|\mathrm{ad}(\xi)\eta\rangle\equiv-\langle B|[\xi,\eta]\rangle\,,$$
$$\forall\xi,\eta\in\mathcal{G}\,, \tag{14}$$

and $\sigma(\xi)$ is the infinitesimal part of $\Sigma(g)$ given explicitly by

[#2] This includes most of the interesting models: $\omega^{ij}(S)=\overset{0}{\omega}{}^{ij}=$const. describes ordinary field theories with S^i denoting both the fields and their canonical momenta; $\omega^{ij}(S)$ linear in S^i describes group coadjoint orbits and $\omega^{ij}(S)$ bilinear in S^i describes W_n-like models.

$$\Sigma(g=\exp\xi)=\sigma(\xi)+\mathrm{O}(\xi^2)\,,$$
$$\sigma(\xi)=-T_i^*\overset{0}{\omega}{}^{ij}\xi_j\,. \qquad (15)$$

In terms of the ordinary coadjoint actions $\mathrm{Ad}^*(g)$ and $\mathrm{ad}^*(\xi)$ of the non-central-extended group G and algebra $\mathscr{G}$, $\Sigma(g)$ in (10) and $\sigma(\xi)$ in (15) are expressed as follows [12]:

$$\Sigma(g)=\mathrm{Ad}^*(g)B_0-B_0+c\lambda S(g)\,,$$
$$\sigma(\xi)=\mathrm{ad}^*(\xi)B_0+c\lambda s(\xi)\,, \qquad (16)$$
$$S(g=\exp\xi)=s(\xi)+\mathrm{O}(\xi^2)\,, \qquad (17)$$

where $S(g)$ is the "integrated anomaly", i.e. the "anomaly" for *finite* group transformations $g\in \mathrm{G}$ due to the presence of the central extension in $\tilde{\mathscr{G}}$ (8). The coefficient λ in (16) is a numerical normalization factor for each specific model.

In terms of $\Sigma(g)$ the fundamental PBs (2) on the $\tilde{\mathrm{G}}$ co-orbit $\mathscr{O}_{(B_0,c)}$ can be rewritten in the form [11,12]

$$\{\langle\Sigma(g)\,|\,\xi\rangle,\langle\Sigma(g)\,|\,\eta\rangle\}_{\mathrm{PB}}$$
$$=\langle\mathrm{ad}^*(\xi)\Sigma(g)+\sigma(\xi)\,|\,\eta\rangle\,,\quad \forall\xi,\eta\in\mathscr{G}\,. \qquad (18)$$

Going back to the general non-linear case (2) and using (5) one can show that the general form of the classical mechanics action on the phase space $\mathscr{M}_S$ corresponding to the PBs (2) is

$$\tilde{W}[S]=W[S]-\int \mathrm{d}t\, H[S]\,, \qquad (19)$$

$$W[S]=-\int\Big\{S^iY_i$$
$$-\tfrac{1}{2}\mathrm{d}^{-1}[(\overset{0}{\omega}{}^{ij}-\tfrac{1}{2}\overset{0}{\omega}{}^{ij}_{kl}S^kS^l)Y_i\wedge Y_j]\Big\}\,, \qquad (20)$$

where H is a hamiltonian on $\mathscr{M}_S$. In (19) and (20) the integral is over an one-dimensional curve on $\mathscr{M}$ with parameter t. Accordingly, the exterior derivative along the curve becomes $\mathrm{d}=\mathrm{d}t\,\partial_t$, and the projection of the one-form Y_i is $Y_i=\mathrm{d}t\,y_i(t)$. Note the presence of the multi-valued term in the "kinetic" part $W[S]$ (20).

In what follows we shall restrict our attention to the purely "kinetic" action (20) (i.e. the case $H[S]=0$), since this is precisely the form of most interesting geometric actions [for instance, the actions on group coadjoint orbits $\mathscr{M}_S=\mathscr{O}_{(B_0,c)}$, see eq. (29)].

Let us consider the following transition on $\mathscr{M}_S$:

$$\delta_\eta S^i=-\omega^{ij}(S)\eta_j\,, \qquad (21)$$

implying that Y_i transform as a "gauge" potential:

$$\delta_\eta Y_i=\mathrm{d}\eta_i+\frac{\delta}{\delta S^i}\omega^{kl}(S)\,Y_k\eta_l$$
$$=\mathrm{d}\eta_i+\overset{0}{\omega}{}^{kl}_i Y_k\eta_l+\overset{0}{\omega}{}^{kl}_{ij}S^jY_k\eta_l\,. \qquad (22)$$

One can easily show, using (22) and (5), that (21) is a Noether symmetry of the action $W[S]$ (20),

$$\delta_\eta W[S]=-\int \mathrm{d}t\, S^i\,\partial_t\eta_i\to\partial_t S^i|_{\text{on-shell}}=0\,, \qquad (23)$$

with $J^i(S)\equiv S^i$ being the corresponding Noether current (actually, a charge).

Functional derivative of $W[S]$ is compactly given by (for arbitrary variations δS^i):

$$\delta W[S]=-\int \delta S^i\, Y_i\,,$$

$$\text{i.e.}\quad \frac{\delta}{\delta S^i(t)}W[S]=-y_i(t)\,. \qquad (24)$$

Therefore, accounting for (5), the action $W[S]$ (20) satisfies the following *off-shell* functional differential equation:

$$\partial_t S^i(t)-\omega^{ij}(S)\frac{\delta}{\delta S^j(t)}W[S]=0\,. \qquad (25)$$

Similarly, the Legendre transform $\Gamma[y]$ of $W[S]$,

$$\Gamma[y]=W[S]+\int \mathrm{d}t\, S^iy_i$$
$$=-\tfrac{1}{2}\int \mathrm{d}^{-1}[(\overset{0}{\omega}{}^{ij}-\tfrac{1}{2}\overset{0}{\omega}{}^{ij}_{kl}S^kS^l)Y_i\wedge Y_j]\,, \qquad (26)$$

$$\frac{\delta}{\delta S^i}W[S]=-y_i\leftrightarrow\frac{\delta}{\delta y_i}\Gamma[y]=S^i\,, \qquad (27)$$

satisfies the functional differential equation

$$\partial_t\left(\frac{\delta}{\delta y_i(t)}\Gamma[y]\right)+\omega^{ij}\left(\frac{\delta}{\delta y}\Gamma[y]\right)y_j(t)=0\,. \qquad (28)$$

Let us stress that in all equations above, involving both S^i and Y_i (or y_i), it is understood that they are functionals of each other determined from the basic off-shell relation (5).

In the important particular case (7), when the phase space $\mathscr{M}_{(B_0,c)}$ is a $\tilde{\mathrm{G}}$ coadjoint orbit $\mathscr{O}_{(B_0,c)}$, formulas (20), (24)–(28) can be rewritten as follows [10–12]:

$$W[S] \equiv W_G[g] = -\int [\langle \Sigma(g) | Y(g) \rangle - \tfrac{1}{2} d^{-1}(\langle \sigma(Y(g)) | Y(g) \rangle)] , \quad (29)$$

$$\frac{\delta}{\delta \Sigma(g)} W_G[g] = -y_t(g) ,$$

$$\frac{\delta}{\delta y_t(g^{-1})} W_G[g] = -\Sigma(g^{-1}) , \quad (30)$$

$$\Gamma_G[y] = W_G[g] + \int \langle \Sigma(g) | Y(g) \rangle = -W_G[g^{-1}] , \quad (31)$$

$$\partial_t \left(\frac{\delta}{\delta y(t)} \Gamma_G[y] \right) - \mathrm{ad}^*(y(t)) \frac{\delta}{\delta y(t)} \Gamma_G[y] - \sigma(y(t)) = 0 . \quad (32)$$

[As in the nonlinear case the one-form $Y(g)$ becomes $Y(g) = dt\, y_t(g)$ along the phase-space curve of integration in the action (29)].

The last equality in eq. (31) is a consequence of the fundamental group composition law [11,12]:

$$W_G[g_1 g_2] = W_G[g_1] + W_G[g_2] + \int \langle \Sigma(g_2) | Y(g_1^{-1}) \rangle \quad (33)$$

generalizing the famous Polyakov–Wiegmann composition laws [13] to the case of geometric actions for arbitrary infinite-dimensional groups $\tilde{G}$ (29).

3. Effective actions and Ward identities

Let us consider the arbitrary classical mechanics model on an infinite-dimensional phase-space $\mathcal{M}_\Phi$ parametrized by coordinates Φ_a (the fundamental Poisson brackets $\{\Phi_a, \Phi_b\}_{PB} = \Omega_{ab}(\Phi)$ need *not* be linear or bilinear with respect to Φ_a; as in sect. 2 the index a labels in general both continuous and discrete indices). Let the classical action $W_0[\Phi]$ posses infinite-dimensional Noether symmetries:

$$\delta_\eta \Phi_a = X_a^i(\Phi) \eta_i ,$$

$$\delta_\eta W_0[\Phi] = -\int J^i(\Phi)\, \partial_t \eta_i . \quad (34)$$

The corresponding Noether conserved currents $J^i(\Phi)$ span the PB algebra of the form

$$\{J^i(\Phi), J^j(\Phi)\}_{PB} = \omega^{ij}(J(\Phi)) , \quad (35)$$

$$\omega^{ij}(J(\Phi)) = \overset{0}{\omega}{}^{ij} + \overset{0}{\omega}{}^{ij}_k J^k(\Phi) + \tfrac{1}{2} \overset{0}{\omega}{}^{ij}_{kl} J^k(\Phi) J^l(\Phi) + \dots , \quad (36)$$

where $\omega^{ij}(J(\Phi))$ has exactly the same properties as $\omega^{ij}(S)$ in (4). Let us recall that $J^i(\Phi)$ are the PB generators of the relevant Noether symmetries:

$$\delta_\eta F(\Phi) = \{\eta_i J^i(\Phi), F(\Phi)\}_{PB} = \frac{\delta F(\Phi)}{\delta \Phi_a} X_a^i(\Phi) \eta_i , \quad (37)$$

for any "observable" $F(\Phi)$. In particular, the transformation of the Noether currents themselves read

$$\delta_\eta J^i(\Phi) = -\omega^{ij}(J(\Phi)) \eta_j . \quad (38)$$

Let us now introduce coupling of the conserved $J^i(\Phi)$ to an external "source" y_i:

$$W_0[\Phi] + \int dt\, J^i(\Phi) y_i . \quad (39)$$

The action (39) is gauge invariant under the Noether transformations (34) provided y_i is simultaneously transformed as a "gauge potential":

$$\delta_\eta y_i = \partial_t \eta_i + \frac{\delta}{\delta J^i} \omega^{kl}(J(\Phi)) y_k \eta_l . \quad (40)$$

In the ordinary Lie-algebra case [$\overset{0}{\omega}{}^{ij} = 0$ and $\overset{0}{\omega}{}^{ij}_{kl} = 0$ in (36)] this is exactly a gauge invariance under the infinite-dimensional group G generated by the Noether currents $J^i(\Phi)$.

Let us consider the quantum effective action:

$$\exp(i\tilde{\Gamma}[y]) \equiv \int \mathcal{D}\Phi_a \exp\left[i\left(W_0[\Phi] + \int dt\, J^i(\Phi) y_i \right)\right] . \quad (41)$$

Performing in (41) change of variables $\Phi^a \to \Phi_a + \delta_\eta \Phi_a \equiv \Phi_a + X_a^i(\Phi)\eta_i$ and using (34) and (38), we get the Ward identity (WI):

$$\partial_t \frac{\delta \tilde{\Gamma}}{\delta y_i(t)} + y_j(t) \omega^{ij}\left(\frac{\delta \tilde{\Gamma}}{\delta y(t)} \right) - \tfrac{1}{2} i y_j(t) \overset{0}{\omega}{}^{ij}_{kl} \frac{\delta^2 \tilde{\Gamma}}{\delta y_k(t) \delta y_l(t)} + R_t^i(y) = 0 . \quad (42)$$

The last term $R_t^i(y)$ in (42) is anomalous and comes from the non-invariance (in general) of the measure in (41) under the above change of variables.

In the non-linear (non Lie-algebra) case of the PB algebra (35) the WI (42) does not form a closed system of functional differential equations for $\tilde{\Gamma}$. Indeed, the second order functional derivative term in (42) corresponds to an insertion of the composite field:

$$\Lambda_t^{ij} = \overset{0}{\omega}{}^{ij}_{k_1 k_2} J_t^{k_1}(\Phi) J_t^{k_2}(\Phi) , \tag{43}$$

with coinciding arguments of the constituents [#3]. Therefore, upon renormalization the second order functional derivative term in (42) describes an insertion of a new composite field which cannot be expressed in terms of the original effective action $\tilde{\Gamma}[y]$.

In what follows we shall only be discussing the Lie-algebra case i.e. we take $\overset{0}{\omega}{}^{ij}_{kl}=0$ in (35) and (36). Then the WI (42) acquires the form

$$\partial_t \frac{\delta \tilde{\Gamma}}{\delta y_i(t)} + y_j(t) \left(\overset{0}{\omega}{}^{ij} + \overset{0}{\omega}{}^{ij}_{k} \frac{\delta \tilde{\Gamma}}{\delta y_k(t)} \right) + R_t^i(y) = 0 , \tag{44}$$

and can be rewritten as

$$\hat{L}_t^i(y)\tilde{\Gamma} + R_t^i(y) + \overset{0}{\omega}{}^{ij} y_j(t) = 0 , \tag{45}$$

where we introduced the following functional differential operator:

$$\hat{L}_t^i(y) \equiv [\delta_j^i \partial_t + \overset{0}{\omega}{}^{ik}_{j} y_k(t)] \frac{\delta}{\delta y_j(t)} , \tag{46}$$

or, in terms of the group notation from (9)–(15)

$$T_i^* \hat{L}_t^i(y) \equiv [\partial_t - \mathrm{ad}^*(y(g))] \frac{\delta}{\delta y(g)} , \tag{47}$$

spanning the algebra

$$[\hat{L}_t^i(y), \hat{L}_{t'}^j(y)] = -\overset{0}{\omega}{}^{ij}_{k} \hat{L}_t^k(y) \delta(t-t') . \tag{48}$$

Using the algebra (48), the Wess–Zumino consistency condition [14] for (45) yields

$$\hat{L}_t^i(y) R_{t'}^j(y) - \hat{L}_{t'}^j R_t^i(y) + \overset{0}{\omega}{}^{ij}_{k} R_t^k(y) \delta(t-t') = 0 . \tag{49}$$

[#3] Note that both the arguments t as well as the continuous parts of the indices $k_{1,2}=(x_{1,2}; A_{1,2})$ of $J_t^{k_{1,2}}(\Phi)$ in (43) coincide since the infinite-dimensional matrix $\overset{0}{\omega}{}^{ij}_{k_1 k_2}$ is an operator kernel of the form $\delta(x_1-x_2)$ and derivatives thereof.

Eq. (49) shows that the "anomaly" $R_t^i(y)$ must be a cohomologically non-trivial solution of the latter [i.e. $R_t^i(y)$ *cannot* be represented in the form $R_t^i(y) = \hat{L}_t^i(y)\Lambda(y)$ with an arbitrary functional $\Lambda(y)$]. One easily checks, using the Jacobi identities in (4), that a cohomologically non-trivial solution to (49) reads

$$R_t^i(y) = r\overset{0}{\omega}{}^{ij} y_j(t) . \tag{50}$$

Furthermore, for the algebras satisfying condition $\dim H^2(\mathscr{G})=1$, this solution is unique up to the numerical factor r to be determined from explicit calculations in each specific model. Substituting (50) into (44) we obtain the renormalized WI:

$$\partial_t \frac{\delta \tilde{\Gamma}}{\delta y_i(t)} + y_i(t) \left((1+r)\overset{0}{\omega}{}^{ij} + \overset{0}{\omega}{}^{ij}_{k} \frac{\delta \tilde{\Gamma}}{\delta y_k(t)} \right) = 0 . \tag{51}$$

Let us now observe that eq. (51), the WI for the quantum effective action $\tilde{\Gamma}$ (41), coincides exactly with the functional differential eq. (28) (with $\overset{0}{\omega}{}^{ij}_{kl}=0$) or, equivalently, eq. (32) for the Legendre-transformed group $\tilde{G}$ co-orbit action Γ_G, where $\tilde{G}$ is the Lie-group corresponding to the PB Lie-algebra $\tilde{\mathscr{G}}$ of the Noether conserved currents (35). Thus, recalling (29)–(31) we get the following main result:

$$\tilde{\Gamma}|_{y \equiv T^i y_i = y(g^{-1})} = -(1+r) W_G[g] . \tag{52}$$

In particular, when the classical Noether algebra (35) $\mathscr{G}$ appears without the cocycle term [i.e. $\overset{0}{\omega}{}^{ij}=0$ in (35), (44), (45), (51)], but admits a central extension, then the solution for the quantum effective action

$$\tilde{\Gamma}|_{y \equiv T^i y_i = y(g^{-1})} = -r W_G[g] \tag{53}$$

is entirely due to the "anomaly" $R_t^i(y)$ in the WI.

Let us particularly stress, that the RHS of (52), (53) does not depend on the details of the classical action $W_0[\Phi]$, but only depends on the structure of its Noether symmetry (34), (35). We have therefore shown, that if the underlying Noether symmetry algebra $\mathscr{G}$ admits a central extension $\tilde{\mathscr{G}}$ and satisfies $\dim H^2(\mathscr{G})=1$, then the central extension yields unique (up to a constant r) solution of the Ward identity for the quantum effective action which takes the form of the geometric coadjoint orbit action of the Noether symmetry group $\tilde{G}$ calculated in terms of the basic group theoretical objects within the symplectic approach [10–12].

4. Examples

4.1. Ward identities in group co-orbit models

Let $\mathcal{M}_\Phi = \mathcal{O}_{(B_0,c)}$ be a co-orbit of the infinite-dimensional group with central extension $\tilde{G}$, and $W[\Phi] = W_G[g]$ be the corresponding co-orbit geometric action [cf. (29)]. As a set of Noether conserved currents we choose $T_i^* J^i(\Phi) = \Sigma(g)$ where $\Sigma(g)$ is the same as in (9). As shown in refs. [11,12,15] $\partial_t \Sigma(g)|_{\text{on-shell}} = 0$ and the PB algebra of $\Sigma(g)$ is (18). Then, our general result (52) tells us that:

$$\int \mathcal{D}\tilde{g} \exp\left[\mathrm{i}\left(W_G[\tilde{g}] + \int \langle \Sigma(\tilde{g}) | y(g^{-1}) \rangle\right)\right]$$

$$= \exp(\mathrm{i}\{-(1+r) W_G[g]\}) . \tag{54}$$

Thus eq. (54) provides the explicit solution of the Ward identities for the generating functional of all correlation functions of the form $\langle \Sigma(g) ... \Sigma(g) \rangle$ in any group coadjoint orbit model.

In particular, let us recall that for $B_0 = 0$ in the case of $\tilde{G}$ – Kac–Moody group (i.e. WZNW models) $\Sigma(g) = \partial_- g g^{-1}$, whereas in the case of $\tilde{G}$ – Virasoro group [i.e. Polyakov $D=2$ gravity] $\Sigma(g) = \mathrm{S}(F)$ where $\mathrm{S}(F)$ is the schwarzian of the Virasoro group element $g \equiv F$ [a conformal diffeomorphism $\tilde{x}^+ = \tilde{x}^+$, $\tilde{x}^- = F(x^+, x^-)$].

4.2. Effective action of toroidal membrane

Let us consider the following semidirect product of infinite-dimensional groups

$$\tilde{G} = \mathrm{Diff}_0(\mathrm{T}^2) \ltimes (\mathcal{HW}) . \tag{55}$$

Here $\mathrm{Diff}_0(\mathrm{T}^2)$ denotes the group of area-preserving diffeomorphisms on torus with the Iliopoulos–Floratos central extension [16] whose Lie algebra reads:

$$[\hat{L}(x), \hat{L}(y)] = -\epsilon^{ij} \partial_i \hat{L}(x) \partial_j \delta^{(2)}(x-y)$$

$$- a^i \partial_i \delta^{(2)}(x-y) . \tag{56}$$

[Here and in what follows $x = (x^1, x^2) \in \mathrm{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$, the indices $i, j = 1, 2$, and the central "charge" a^i is a numerical two-vector.] The group elements of $\mathrm{Diff}_0(\mathrm{T}^2)$ are smooth diffeomorphisms on the torus $x^i \to F^i = F^i(x^1, x^2)$ subject to the area-preserving constraint:

$$\epsilon^{ij} \partial_i F^k \partial_j F^l = \epsilon^{kl} . \tag{57}$$

The second factor in the semidirect product (55) $(\mathcal{HW})$ denotes the Heisenberg–Weyl group with the Lie algebra

$$[\hat{X}^I(x), \hat{P}^J(y)] = \delta^{IJ} \delta^{(2)}(x-y) , \tag{58}$$

and group elements of the form

$$\exp\left(\int \mathrm{d}^2 x \, [P_I(x) \hat{X}^I(x) + X_I(x) \hat{P}^I(x)]\right),$$

where $I = 1, ..., N$.

Applying the general formalism of refs. [10,12] we get the following geometric action on the adjoint orbit of $\tilde{G}$ (55) [we take for simplicity the orbit with initial point $(B_0, c) = (0, c)$; cf. (10), (15)]:

$$W_{\tilde{G}} = W_{\mathrm{Diff}_0(\mathrm{T}^2)}[F] + W[X, P; y(F^{-1})] , \tag{59}$$

$$W_{\mathrm{Diff}_0(\mathrm{T}^2)}[F]$$

$$= -\tfrac{1}{3} \int \mathrm{d}t \, \mathrm{d}^2 x \, (a^k \epsilon_{kl} F^l) \epsilon_{ij} F^i \partial_t F^j , \tag{60}$$

$$W[X, P; y(F^{-1})] = \int \mathrm{d}t \, \mathrm{d}^2 x \, [P^I \partial_t X_I$$

$$- y(F^{-1}) \epsilon^{ij} \partial_i X^I \partial_j P_I] . \tag{61}$$

The first part (60) of the action (59) is the geometric co-orbit action for the pure $\mathrm{Diff}_0(\mathrm{T}^2)$ obtained in ref. [15]. The second part (61) describes the "coupling" of the "matter" fields X^I, P^I to the "gauge" field $y(F^{-1})$. $Y(F) \doteq \mathrm{d}t \, y(F)$ denotes the basic Maurer–Cartan one-form [cf. (12), (13)] for the group $\mathrm{Diff}_0(\mathrm{T}^2)$ (F^{-1} indicating the inverse area-preserving diffeomorphism). The explicit form of $Y(F)$ found in ref. [15] reads

$$Y(F) = \tfrac{1}{2} \epsilon_{ij} F^i \, \mathrm{d}F^j + \mathrm{d}\rho(F) , \tag{62}$$

where the function $\rho(F)$ is a solution of the consistent [due to (57)] overdetermined system

$$\partial_i \rho(F) = -\tfrac{1}{2} (\epsilon_{kl} F^k \partial_i F^l + \epsilon_{ij} x^j) . \tag{63}$$

Let us now recall, that the classical action of the membrane in the light-cone gauge reads [17,4,16]:

$$W_{\text{membrane}} = \int \mathrm{d}t\, \mathrm{d}^2x\, [P^I\, \partial_t X_I - (P_I P^I + \tfrac{1}{2}\{X_I, X_J\}\{X^I, X^J\}) - \Lambda \epsilon^{ij}\, \partial_i X^I\, \partial_j P_I] \equiv W[X, P; \Lambda] - \int \mathrm{d}t\, \mathrm{d}^2x\, (P_I P^I + \tfrac{1}{2}\{X_I, X_J\}\{X^I, X^J\}) , \qquad (64)$$

where the brackets { , } indicate two-dimensional PBs: $\{X^I, X^J\} = \epsilon^{kl}\, \partial_k X^I\, \partial_l X^J$. The action (64) has precisely the form of a group co-orbit action for (55) with vanishing Iliopoulos–Floratos central charge and with a non-zero hamiltonian [cf. eq. (19)], provided the Lagrange multiplier Λ for the first-class constraints $\epsilon^{ij}\, \partial_i X^I\, \partial_j P_I = 0$ [#4] is parametrized in terms of the group parameters F of $\mathrm{Diff}_0(\mathrm{T}^2)$ as (cf. (62)):

$$\Lambda = y(F^{-1}) = \tfrac{1}{2}\epsilon_{ij}(F^{-1})^i\, \partial_t (F^{-1})^j + \partial_t \rho(F^{-1}) . \qquad (65)$$

Now, let us consider the quantum effective action of the membrane:

$$\exp(\mathrm{i}\tilde{\Gamma}_{\text{membrane}}[\Lambda]) = \int \mathscr{D}X^I\, \mathscr{D}P^I \exp\Big[\mathrm{i}\Big(\int \mathrm{d}t\, \mathrm{d}^2x\, [P^I\, \partial_t X_I - (P_I P^I + \tfrac{1}{2}\{X_I, X_J\}\{X^I, X^J\}) - \Lambda \epsilon^{ij}\, \partial_i X^I\, \partial_j P_I]\Big)\Big] . \qquad (66)$$

Clearly, Λ plays the role of external "source" coupled to the conserved currents $J(X, P) \equiv \epsilon^{ij}\, \partial_i X^I\, \partial_j P_I$, whose PB algebra exactly coincides with the Lie algebra (56) of $\mathrm{Diff}_0(\mathrm{T}^2)$ (with zero central charge). Thus, applying the general result eq. (53) we find

$$\tilde{\Gamma}_{\text{membrane}}[\Lambda = y(F^{-1})] = -W_{\mathrm{Diff}_0(\mathrm{T}^2)}[F] . \qquad (67)$$

The specific value of the induced central charge a^i in the anomalous membrane effective action (67) [see eq. (60)] has to be found from explicit calculations. The principal problem in this context is to find an appropriate regularization (e.g., it is not clear so far that the "discrete" $\mathrm{SU}(\mathscr{N})$-regularization [17,18] will work for this purpose). This question deserves a careful study [#5].

[#4] Recall, that unlike the string, in the case of membranes the light-cone gauge only partially fixes the gauge symmetries leaving the group of area-preserving diffeomorphisms as a residual gauge symmetry; see e.g. refs. [17,4,16].

[#5] In terms of usual massless scalar fields in $D = 2+1$ it is possible to construct unitary, but *not* highest-weight, representations of the area-preserving diffeomorphisms algebra (56) with a *zero* central charge [19]. For a recent progress in constructing unitary highest weight representations of infinite-dimensional algebras in $D \geqslant 3$, see ref. [20].

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